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LETTER TO THE EDITOR

Casimir operators for $\mathrm{su}_q(n)$

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Abstract. Explicit simple expressions are given for Casimir operators of q-deformed $\mathrm{su}_q(n)$. These expressions are similar to the corresponding Casimir operators of non-deformed $\mathrm{su}(n)$. The proof of invariance is based on direct applications of the commutation relations that define $\mathrm{su}_q(n)$ written in a convenient basis.

In recent years there has been considerable interest in the so-called quantum groups $\mathrm{U}_q(g)$, which are q-deformations of the universal enveloping algebra of the Lie algebra g. In this paper we concentrate on the case $g=\mathrm{su}(n)$ but note that the other classical algebras (orthogonal and symplectic) can be treated in much the same way.

As is well known, the non-deformed $\mathrm{su}(n)$ can be defined by generators A_{ab} obeying the commutation relations

$$[A_{ab}, A_{cd}]=\delta_{bc}A_{ad}-\delta_{ad}A_{cb} \tag{1}$$

and the constraint

$$A_{aa}=0. \tag{2}$$

Here all indices range from 1 to n and summation over repeated indices is understood. Further

$$A^{\dagger}_{ab}=A_{ba}. \tag{3}$$

Defining recursively powers of the generators by $(k=1, 2, \ldots)$

$$(A^k)_{ab}=(A^{k-1})_{ac}A_{cb} \qquad (A^0)_{ac}=\delta_{ac} \tag{4}$$

one readily finds

$$[A_{ab}, (A^k)_{cd}]=\delta_{bc}(A^k)_{ad}-\delta_{ad}(A^k)_{cb}. \tag{5}$$

Consequently the objects

$$C_k\equiv(A^k)_{aa} \tag{6}$$

obey

$$[A_{ab}, C_k]=0. \tag{7}$$

Thus the C_k are Casimir operators of kth degree of $\mathrm{su}(n)$.

The object of this letter is the determination of Casimir operators of $\mathrm{su}_q(n)$. Recently Zhang *et al* [1, 2] have described a method for constructing Casimir invariants starting from the Drinfeld [3] universal R-matrices of quantum groups. We follow a more direct approach, parallelling the approach described above for non-deformed su(n). Explicit expressions for the quadratic and cubic Casimirs of $su_q(n)$ were given by Chakrabarti [4], based on the work of Pasquier and Saleur [5]. Our general results for Casimirs of any degree, when specilialized to degree two and three, agree with Chakrabarti.

Next we describe a basis for $su_q(n)$, which is a slight modification of a basis given by Jimbo [6] and in terms of which extremely simple expressions result for the Casimir operators. Then we give these expressions for the Casimir operators and prove their invariance.

Jimbo [6] defines the quantum group $su_q(n)$ in terns of the Cartan elements h_i, the simple raising elements e_i and simple lowering elements f_i ($1 \leqslant i \leqslant n-1$) that obey

$$[h_i, h_j]=0 \tag{8}$$

$$[h_i, e_j]=a_{ij}e_j \qquad [h_i, f_j]=-a_{ij}f_j \tag{9}$$

$$[e_i, f_j]=\delta_{ij}\frac{q^{h_i}-q^{-h_i}}{q-q^{-1}} \tag{10}$$

$$e_ie_j=e_je_i \qquad f_if_j=f_jf_i \qquad (|i-j|\geqslant 2) \tag{11}$$

$$e_i^2e_j-(q+q^{-1})e_ie_je_i+e_je_i^2=0 \qquad |i-j|=1 \tag{12}$$

$$f_i^2f_j-(q+q^{-1})f_if_jf_i+f_jf_i^2=0 \qquad |i-j|=1 \tag{13}$$

and a_{ij} is the Cartan matrix

$$a_{ij}=\begin{cases} 2 & i-j=0 \\ -1 & |i-j|=1 \\ 0 & |i-j|\geqslant 2. \end{cases} \tag{14}$$

Further we may take without loss of generality

$$f_i=e_i^\dagger \qquad h_i=h_i^\dagger. \tag{15}$$

We now observe that a more convenient notation, which brings out the parallelism with non-deformed su(n), is as follows:

Let

$$h_a\equiv\varepsilon_a-\varepsilon_{a+1} \qquad \varepsilon_1+\varepsilon_2+\ldots+\varepsilon_n=0 \tag{16}$$

$$E_{ab}\equiv\begin{cases} E_{ac}E_{cb}-q^{-1}E_{cb}E_{ac} & a<c<b \text{ (no summation)} \\ e_a & a+1=b \\ (q-q^{-1})^{-1} & a=b \\ 0 & a>b \end{cases} \tag{17}$$

$$E_{ab}^\dagger\equiv F_{ba} \qquad \varepsilon_a^\dagger=\varepsilon_a \qquad q^\dagger=q. \tag{18}$$

Now all indices range from 1 to n; note that the index c in (17) is arbitrary as long as $a<c<b$.

In terms of these symbols all of the equations (8)-(15) are summarized as

$$[E_{kk+1}, q^{\varepsilon_a}E_{ab}q^{\varepsilon_b}]=\delta_{k+1a}q^{\varepsilon_a}E_{kb}q^{\varepsilon_b}-\delta_{kb}q^{\varepsilon_a}E_{ak+1}q^{\varepsilon_b} \tag{19}$$

$$[E_{kk+1}, F_{ba}]=\delta_{k+1b}F_{ka}q^{\varepsilon_k-\varepsilon_{k+1}}-\delta_{ka}q^{-\varepsilon_k+\varepsilon_{k+1}}F_{bk+1} \tag{20}$$

and the Hermitian conjugates of (19) and (20).

Our definition of 'Hermitian conjugate' implies that q is real. The formalism could be modified to accommodate complex values of q but for simplicity we assume from now on that q is real.

We remark that (8)–(15) are invariant under $q \leftrightarrow q^{-1}$ whereas (17) and (18) are not. Indeed we can define *another* set of $\tilde{E}_{ab}$ and $\tilde{F}_{ab}$ by

$$\tilde{E}_{ab} = QE_{ab} \qquad \tilde{F}_{ab} = QF_{ab} \tag{21}$$

where Q means $q \leftrightarrow q^{-1}$. The commutation relations obeyed by these objects follow by applying Q to (19) and (20).

Define

$$M_{ab} \equiv E_{ac} q^{\varepsilon_a - 2a} F_{cb} q^{\varepsilon_c + 2c}. \tag{22}$$

Then

$$C_{qp} \equiv q^{2a} (M^p)_{aa} \tag{23}$$

are the desired Casimir operators of $su_q(n)$ with the pth power defined recursively by $(p = 1, 2, \ldots)$

$$(M^{p+1})_{ab} = (M^p)_{ac} M_{cb}. \tag{24}$$

The first step in the proof of invariance of the C_{2p} is the demonstration that

$$[E_{kk+1}, M_{ab}] = \delta_{k+1a} q^{\varepsilon_{k+1} - \varepsilon_k - 1} M_{kb} - \delta_{kb} M_{ak+1} q^{\varepsilon_{k+1} - \varepsilon_k - 1}. \tag{25}$$

The derivation of (25) is left to the appendix.

Given (25) it follows by induction on p that

$$[E_{kk+1}, (M^p)_{ab}] = \delta_{k+1a} q^{\varepsilon_{k+1} - \varepsilon_k - 1} (M^p)_{kb} - \delta_{kb} (M^p)_{ak+1} q^{\varepsilon_{k+1} - \varepsilon_k - 1} \tag{26}$$

and therefore

$$\begin{aligned} [E_{kk+1}, C_{2p}] &= [E_{kk+1}, q^a (M^p)_{aa}] \\ &= \delta_{k+1a} q^{2a + \varepsilon_{k+1} - \varepsilon_k - 1} (M^p)_{ka} - \delta_{ka} (M^p)_{ak+1} q^{2a + \varepsilon_{k+1} - \varepsilon_k - 1} \\ &= q^{2k+1+\varepsilon_{k+1} - \varepsilon_k} (M^p)_{kk+1} \sum_{a=1}^{n} (\delta_{k+1a} - \delta_{ka}) = 0 \end{aligned} \tag{27}$$

since

$$\sum_{a=1}^{n} (\delta_{k+1a} - \delta_{ka}) = \delta_{k+11} - \delta_{kn} = 0 \tag{28}$$

since $1 \leqslant k \leqslant n-1$.

Next we observe that

$$M^{\dagger}_{ab} = q^{\varepsilon_a - \varepsilon_b - 2(a-b) + 1 - \delta_{ab}} M_{ba} \tag{29}$$

and therefore

$$(M^p)^{\dagger}_{aa} = (M^p)_{aa}. \tag{30}$$

Hence by taking the Hermitian conjugate of (27) we find

$$[F_{k+1k}, C_{2p}] = 0. \tag{31}$$

Lastly

$$[\varepsilon_k, C_{2p}] = 0. \tag{32}$$

This is obvious since the ε_k are Cartan generators and therefore commute among themselves and with any function of raising and lowering generators with saturated subscripts.

This completes the proof of invariance of the C_{2p}.

We next observe that by the same argument we have invariance of

$$\tilde{C}_{2p} \equiv QC_{2p} = q^{-2a}(\tilde{M}^p)_{aa} \tag{33}$$

with

$$\tilde{M}_{ab} = QM_{ab} = \tilde{E}_{ac} q^{-\varepsilon_a+2a} \tilde{F}_{cb} q^{-\varepsilon_c-2c}. \tag{34}$$

Clearly, if desired, we may consider in place of C_{2p} and $\tilde{C}_{2p}$ the following even functions of $q-q^{-1}$:

$$\mathscr{C}_{2p} \equiv (C_{2p}+\tilde{C}_{2p})/(q+q^{-1}) \tag{35}$$

and

$$\mathscr{C}_{2p+1} \equiv (C_{2p}-\tilde{C}_{2p})/(q-q^{-1}). \tag{36}$$

The $\mathscr{C}_k$, $k=2,3,\ldots$, beside being invariant under $q \leftrightarrow q^{-1}$ turn out to be more convenient for comparison with the Casimir operators of non-deformed su(n).

Setting in the limit $q \to 1$

$$\varepsilon_a \to A_{aa} \text{ (no summation)} \qquad 1 \leqslant a \leqslant n \tag{37}$$

$$E_{ab} \to A_{ab} \qquad 1 \leqslant a < b \leqslant n \tag{38}$$

$$F_{ba} \to A_{ba} \qquad 1 \leqslant a < b \leqslant n \tag{39}$$

we see that the relations (16)–(20) defininl su$_q(n)$ go over into the relations (1)–(3) defining su(n).

However since

$$E_{aa} = F_{aa} = (q-q^{-1})^{-1} \qquad \text{(no summation)} \tag{40}$$

$$\tilde{E}_{aa} = \tilde{F}_{aa} = -(q-q^{-1})^{-1} \qquad \text{(no summation)} \tag{41}$$

our expressions for $\mathscr{C}_k$ contain terms that become infinite in the $q \to 1$ limit.

Consider, for example, C_2. The divergent part of C_2 is $\Sigma_a\, q^{2(\varepsilon_a+a)}/(q-q^{-1})^2$, hence clearly the following expression

$$C_2 - (q^{2n+1}-q)(q-q^{-1})^{-3} \tag{42}$$

is just as good a Casimir operator as C_2 but it is finite as $q \to 1$.

Consequently the expressions

$$\mathscr{C}_2 - \frac{q^{2n+1}-q-q^{-2n-1}+q^{-1}}{(q^2-q^{-2})(q-q^{-1})^2} \tag{43}$$

$$\mathscr{C}_3 - \frac{q^{2n+1}-q+q^{-2n-1}-q^{-1}}{(q-q^{-1})^4} \tag{44}$$

could be used in place of $\mathscr{C}_2$ and $\mathscr{C}_3$—in the $q \to 1$ limit they reduce precisely to the quadratic and cubic Casimirs of non-deformed su(n).

In general the $\mathscr{C}_k$, modified by appropriate functions of $\mathscr{C}_{k'}$ with $k' < k$, reduce precisely in the $q \to 1$ limit to Casimir operators of degree k of non-deformed su(n).

For this reason we believe that the $\mathscr{C}_k$, $k=2,3,\ldots,n$, defined by (35) and (36) provide a basis for Casimir operators for $su_q(n)$.

In closing we remark that an alternate set of Casimir operators is given by

$$\mathscr{C}'_{2p}=(C'_{2p}+\tilde{C}'_{2p})/(q+q^{-1}) \tag{45}$$

$$\mathscr{C}'_{2p+1}=(C'_{2p}-\tilde{C}'_{2p})/(q-q^{-1}) \tag{46}$$

where

$$C'_{2p}=q^{-2a}(N^p)_{aa} \tag{47}$$

$$N_{ab}=F_{ac}q^{\varepsilon_a+2a}E_{cb}q^{\varepsilon_c-2c}. \tag{48}$$

The proof of invariance of the $\mathscr{C}'_k$ starts from showing that

$$[F_{k+1k},N_{ab}]=\delta_{ka}q^{\varepsilon_k-\varepsilon_{k+1}-1}N_{k+1b}-\delta_{k+1b}N_{ab}q^{\varepsilon_k-\varepsilon_{k+1}-1} \tag{49}$$

and proceeds from there in just the same way as for the $\mathscr{C}_k$.

We have described a procedure for the determination of Casimir operators of $su_q(n)$ which parallels the approach used for non-deformed $su(n)$.

We define $su_q(n)$ in terms of the Cartan elements h_i and the simple raising and lowering elements e_i and f_i and their commutation relations expressed in a convenient basis.

We then show that certain functions of the e_i, f_i and h_i, in fact polynomials in e_i, f_i and $\exp(\pm h_i)$, commute with all the e_i, f_i, h_i and are therefore Casimir operators.

This approach makes no use of the co-multiplication and the Hopf algebra aspect of quantum groups. We view that as an advantage of this formulation. However, it has been pointed out to us by the referee that our matrix M_{ab}, equation (22), appears to be identical to the matrix

$$M[(\pi\otimes I)R^{\mathrm{T}}][(\pi\otimes I)R]$$

where R is the universal R-matrix and π the vector irrep. Using this connection our invariants can be related to those of Zhang *et al* [1, 2].

Appendix. Derivation of equation (25)

We rewrite (22) as

$$M_{ab}=q^{\varepsilon_c}E_{ac}q^{\varepsilon_a}F_{cb}q^{2c-2a+\delta_{bc}-\delta_{ac}} \tag{A1}$$

and use (19) and (20) to obtain

$$\begin{aligned}[E_{kk+1},M_{ab}]&=\{[E_{kk+1},q^{\varepsilon_c}E_{ac}q^{\varepsilon_a}]F_{cb}+q^{\varepsilon_c}E_{ac}q^{\varepsilon_a}[E_{kk+1},F_{cb}]\}q^{2c-2a+\delta_{bc}-\delta_{ac}}\\ &=\{(\delta_{k+1a}q^{\varepsilon_c}E_{kc}q^{\varepsilon_a}-\delta_{kc}q^{\varepsilon_c}E_{ak+1}q^{\varepsilon_a})F_{cb}\\ &\quad+q^{\varepsilon_c}E_{ac}q^{\varepsilon_a}(1-\delta_{cb})(\delta_{k+1c}F_{kb}q^{\varepsilon_k-\varepsilon_{k+1}}-\delta_{kb}q^{-\varepsilon_k+\varepsilon_{k+1}}F_{ck+1})\}q^{2c-2a+\delta_{bc}-\delta_{ac}}.\end{aligned} \tag{A2}$$

Consider the last term in the above:

$$\begin{aligned}&-\delta_{kb}(1-\delta_{cb})q^{\varepsilon_c}E_{ac}q^{\varepsilon_a-\varepsilon_k+\varepsilon_{k+1}}F_{ck+1}q^{2c-2a+\delta_{bc}-\delta_{ac}}\\ &\qquad=-\delta_{kb}(1-\delta_{ck})q^{\varepsilon_c}E_{ac}q^{\varepsilon_a}F_{ck+1}q^{2c-2a+\delta_{k+1c}-\delta_{ac}-\varepsilon_k+\varepsilon_{k+1}-1}\\ &\qquad=-\delta_{kb}M_{ak+1}q^{-\varepsilon_k+\varepsilon_{k+1}-1}.\end{aligned} \tag{A3}$$

To get this result we used the fact that we must sum over c from $\max(a, b)$ to n. This follows from the definition of M_{ab} since $E_{ac}=0$ for $a>c$ and $F_{cb}=0$ for $b>c$. But the factor δ_{kb} changes $\max(a, b)$ to $\max(a, k)$ and the factor $1-\delta_{ck}$ eliminates $c=k$ from the sum leaving the range $\max(a, k+1) \leqslant c \leqslant n$, which is precisely the correct range for the definition of M_{ak+1}.

Next we evaluate the first term in (A2):

$$\begin{aligned} \delta_{k+1a} q^{\varepsilon_c} E_{kc} q^{\varepsilon_a} F_{cb} q^{2c-2a+\delta_{bc}-\delta_{ac}} & \\ &= \delta_{k+1a} q^{-\varepsilon_k+\varepsilon_{k+1}-1+\varepsilon_c} E_{kc} q^{\varepsilon_k} F_{cb} q^{2c-2k+\delta_{bc}-\delta_{kc}} \\ &= \delta_{k+1a}\{q^{-\varepsilon_k+\varepsilon_{k+1}-1} M_{kb} - (q-q^{-1})^{-1} q^{\varepsilon_{k+1}} F_{kb} q^{\varepsilon_k-1}\}. \qquad \text{(A4)} \end{aligned}$$

Lastly the second and third terms in (A2) give

$$\begin{aligned} -\delta_{kc} q^{\varepsilon_k} E_{a\,k+1} q^{\varepsilon_a} F_{kb} q^{2k-2a+\delta_{bk}-\delta_{ak}} & \\ &\quad + \delta_{k+1c}(1-\delta_{k+1b}) q^{\varepsilon_{k+1}} E_{a\,k+1} q^{\varepsilon_a} F_{kb} q^{\varepsilon_k-\varepsilon_{k+1}+2k+2-2a+\delta_{bk+1}-\delta_{ak+1}} \\ &= q^{\varepsilon_k} E_{a\,k+1} q^{\varepsilon_a} F_{kb} q^{2k-2a+\delta_{bk}-\delta_{ak}} \sum_{c=\max(a,b)}^{n} [\delta_{k+1c}(1-\delta_{k+1b}) - \delta_{kc})] \\ &= \delta_{k+1a}(q-q^{-1})^{-1} q^{\varepsilon_{k+1}} F_{kb} q^{\varepsilon_k-1}. \qquad \text{(A5)} \end{aligned}$$

Adding (A3), (A4) and (A5) yields (25) in the text.

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